Matrices and Linear Equations

Goal: prove dimension is well-defined for finite dimensional vector spaces.

Example 1: Solve

$$3x - 8y = 6$$
$$-x + 7y = -2$$

for x and y.

Could do this via brute force, but instead...

Matrix form:

$$A = \begin{bmatrix} 3 & -8 \\ -1 & 7 \end{bmatrix},$$

the coefficient matrix.

$$b = \begin{bmatrix} 6 \\ -2 \end{bmatrix}.$$ Consider the

augmented matrix

$$\begin{bmatrix} A \mid b \end{bmatrix} = \begin{bmatrix} 3 & -8 & 6 \\ -1 & 7 & -2 \end{bmatrix}$$

Row reduction:

The system of equations we started with, if it has solutions, has solutions invariant under

1) multiplying one equation by a constant

2) adding a multiple of one equation to another.

Row reduction is the process of carrying out these operations on either the coefficient or augmented matrix.

Let's do augmented:

$$\begin{bmatrix} 3 & -8 & 6 \\ -1 & 7 & -2 \end{bmatrix}$$

swap R1 with R2

$$\begin{bmatrix} -1 & 7 & -2 \\ 3 & -8 & 6 \end{bmatrix}$$

add 3R1 to R2

$$\begin{bmatrix} -1 & 7 & -2 \\ 0 & 13 & 0 \end{bmatrix}$$

So $y=0$ and $x=2$.

Can check:

$$\begin{bmatrix} -1 & 7 \\ 3 & -8 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$$

$$= b \checkmark$$

Definition: (echelon form)

A matrix B is in echelon form if, through a sequence of row operations as in example 1, we have

1) Any rows with all zeros in the entries are the bottom rows of the matrix.

2) The first nonzero entry of any row is in a column strictly to the right of the row above it.

Pivots: The pivots are the nonzero row entries referenced in the second part of the definition of echelon form. For example:

pivot

$$\begin{bmatrix} -1 & 7 & -2 \\ 0 & 13 & 0 \end{bmatrix}$$

pivot

Observation: If $B \in M_{n\times m}(\mathbb{C})$, the number of pivots in the echelon form of B is less than or equal to $k = \min(n, m)$. So the number of pivots is less than or equal to the number of rows or columns.

Example 2: (using Wolfram Alpha)

$$3x + 4y + 8z = 5$$

$$-x - y + 3z = 10$$

$$14x + 17y - 3z = 12$$

$$56x + 49y + 72z = -93$$

Augmented matrix

$$\begin{bmatrix} 3 & 4 & 8 & 5 \\ -1 & -1 & 3 & 10 \\ 14 & 17 & -3 & 12 \\ 56 & 49 & 72 & -93 \end{bmatrix}$$

(reduced) echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivots circled

Example 3: (Wolfram Alpha)

$$5x_1 - 3x_2 + 7x_3 - 8x_4 + x_5 = 0$$

$$8x_1 + 3x_2 + 13x_3 - 2x_4 + 2x_5 = 11$$

Augmented matrix

$$\begin{bmatrix} 5 & -3 & 7 & -8 & 1 & 0 \\ 8 & 3 & 13 & -2 & 2 & 11 \end{bmatrix}$$

(reduced) echelon form

$$\begin{bmatrix} 1 & 0 & \frac{20}{13} & -\frac{10}{13} & \frac{3}{13} & \frac{11}{13} \\ 0 & 1 & 3/13 & 18/13 & 2/39 & 55/39 \end{bmatrix}$$

pivots circled